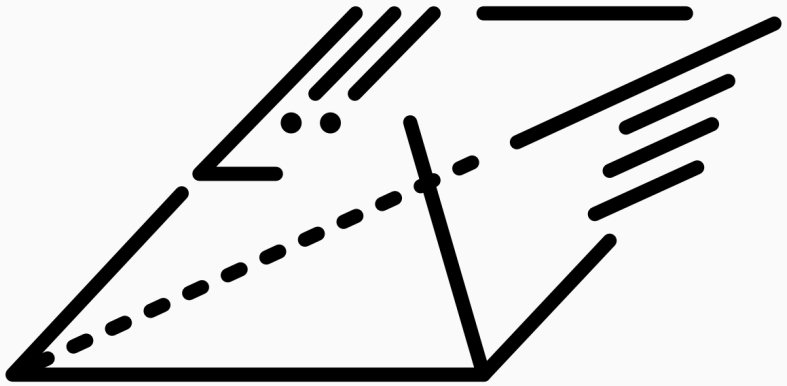


sHoTT working group

Recap

Fredrik Bakke · *Ulrik Buchholtz* · ***Sam Toth*** · *Jonathan Weinberger* ·
Reid Barton · Evan Cavallo · Håkon Gylterud · Nima Rasekh · Emily Riehl ·
Tashi Walde · (Mark Williams)

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What we had

What we had

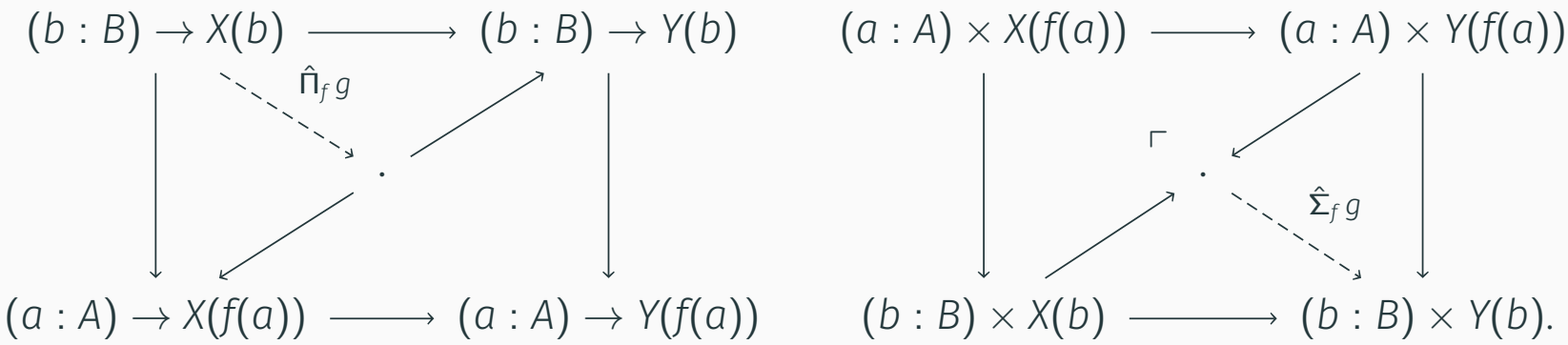
Foundations

- Infrastructure for lifting properties
- Colimits & limits (modified join, free localisations, splitting construction)
- Algebra (quasicoherence)

Categories

- Segal types and variants
- Yoneda lemma & covariant families
- Algebraic laws
- Spine and necklace lifting
- The classifier of spaces

Lifting infrastructure — Leibniz Adjunction



Theorem (Dependent Leibniz adjunction)

For every triple of maps $f : A \to B$, $g : (b : B) \to X(b) \to Y(b)$, and $h : (b : B)\,(y : Y(b)) \to U(b)(y) \to V(b)(y)$ there is an equivalence of arrows

$$\left(\hat{\Pi}_{\hat{\Sigma}_f\, g}((b, y) \mapsto h(b)(y))\right) \simeq \left(\hat{\Pi}_f(b \mapsto \hat{\Pi}_{g(b)}\, h(b))\right).$$

Closure properties of orthogonality

```
open gSubU Category-gSubU public using () renaming
  ( ∈S←equiv to is-category←equiv)
open gReflectiveClosure Category-gRSubU public using () renaming
  ( Π∈S to Π-is-category
  ; ULift∈S to ULift-is-category
  ; 𝟙∈S to 𝟙-is-category
  ; ∈S←is-singleton to is-category←is-singleton
  ; ∈S←retract to is-category←retract
  ; Pullback∈S to Pullback-is-category
  ; fibre∈S to fibre-is-category
  ; Ext∈S to Ext-is-category
  ; ×∈S to ×-is-category
  ; =∈S to =-is-category
  ; ~∈S to ~-is-category
  ; retract∈S to retract-is-category
  ; biinv∈S to biinv-is-category
  ; ≃∈S to ≃-is-category
  ; Equaliser∈S to Equaliser-is-category
  ; Seq-lim∈S to Seq-lim-is-category
  ; preidempotent-map∈S to preidempotent-map-is-category
  ; qidempotent-map∈S to qidempotent-map-is-category
  ; idempotent-map∈S to idempotent-map-is-category
  ; Retracts∈S to Retracts-is-category)
open gRSubUHomClosure Category-gRSubU public using () renaming
  ( Hom∈S to Hom-is-category)
```

Category theory

- Segal* types have unital and associative composition that satisfy the triangle and pentagon laws.
- Theory of left fibrations, and (left/right) Segal types satisfy the Yoneda lemma.
- Every inclusion $\Delta^1 \vee \ldots \vee \Delta^1 \hookrightarrow \Delta^n$ is inner.
- Every inclusion $\Delta^{n_1} \vee \ldots \vee \Delta^{n_k} \hookrightarrow \Delta^{\tilde{n}}$ is inner.
- There is a classifying object for left fibrations $\mathcal{S}$.

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What we did and why

What we did

- Installed Agda!
- $\sim$12 PRs
- Nima showed that synthetic quasicoherence for $\mathbb{I}[x]$ is equivalent to the generalised Phoa's principle for $\mathbb{I}[x]$, and that they imply every endofunction on $\mathbb{I}$ is monotone.
- Fredrik worked on infrastructure for Algebra
- Reid showed that if every endofunction on $\mathbb{I}$ is monotone, the core of $\mathbb{I}$ has two elements, and cubes separate, then the conjunctive property holds.
- Evan did lattice structure for Δ^n, working towards showing it is a category.
- Håkon proved parametricity for Space, and worked on an application for polynomial endofunctors
- Emily constructed full subtypes, and proved their universal property
- Tashi worked on localisations of categories.
- Sam worked on Spaces being Segal

Algebra

Definition
There is an adjunction $\mathcal{O} \dashv \mathrm{Spec}$ between Set and $\mathbb{I}\text{-}\mathrm{alg}^{op}$. An $\mathbb{I}$-algebra A is quasi-coherence if the counit map $\mathsf{Hom}_{\mathbb{I}}(A, \mathcal{O}(\mathrm{Spec}(A))$ is an equivalence.

- The cubical spaces model validates that every finitely presented $\mathbb{I}$-algebra is quasi-coherent

Theorem (Formalised by Nima)
Quasicoherence of $\mathbb{I}[X]$ is equivalent to Phoa's principle for $\mathbb{I}[X]$. That every map $f : \mathbb{I} \to \mathbb{I}$ is equal to $i \mapsto f(0) \vee (i \wedge f(1))$ and hence monotone.

Construction (Formalised by Evan)
There is a lattice structure on Δ^n for every n. Work in progress on generalising the fact that $\mathbb{I}$ arrises as the spectrum of an $\mathbb{I}$-algebra ($\mathrm{Spec}(\mathbb{I}[X]) \simeq \mathbb{I}$).

$\rightsquigarrow$ showing Δ^n is a category

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More Algebra

Definition
We say *cubes separate* if every map $f : A \to B$ is an equivalence if every $n : \mathbb{N}$ the map $\flat(\mathbb{I}^n \to A) \to \flat(\mathbb{I}^n \to B)$ is an equivalence.

Theorem (Formalised by Reid)
Assuming all endomaps on $\mathbb{I}$ are monotone, $\mathbb{I}$ has two global points, and cubes separate, then for any two $i, j : \mathbb{I}$, if $(i \wedge j) = 0$, then $i = 0$ or $j = 0$.

$\rightsquigarrow$ properties of $\mathcal{S}$.

Parametricity

Definition (Lord)
A universe $\mathcal{U}$ is *parametric* if, for every $X : \mathcal{U}$, every map $\mathcal{U} \to X$ is uniquely constant.

Theorem (Formalised by Håkon)
The universe of spaces is parametric.

Corollary (New formalisation targets)

- *There is a unique family of functions* $(X : \mathcal{S}) \to X \to X$.
- *Given* $F, G : \mathcal{S} \to \mathcal{S}$*, then every family of functions* $(X : \mathcal{S}) \to F(X) \to G(X)$ *is natural.*

TODO: Applications to polynomial endofunctors

Full subcategories

- We use **modalities** in Agda to allow non-functorial dependency ($\flat \dashv \sharp$)
- We may suppose the the $\flat$ discrete types are exactly the Δ^1-null types. $\flat A \leftrightsquigarrow A^{\simeq}$
- A subtype of $\flat A$ then corresponds to a subgroupoid of the core of A

Construction (Formalised by Emily)
If ϕ is a subtype of $\flat A$ then we call $A_\phi := \Sigma_{a:A} \langle \sharp | \phi(a) \rangle$ the full subtype of A determined by ϕ. The map has the universal property that $A_\phi \to A$ is orthogonal to every $\flat X \to X$.

Corollary
Supposing that $\partial\Delta^1 \to \flat\Delta^1$ is an equivalence, $A_\phi \to A$ is fully-faithful.

Proof.
$\partial\Delta^1 \to \flat\Delta^1 \to \Delta^1$. □

TODO: More full subcategories and localisation

- Still need to show that if A is a category, then so is A_ϕ
- Need to think about non-full subcategories
- HoTT foundations gives us localisations at types (or maps) — we can localise at all morphisms in a type. Standard construction of localisations require non-full subcategories

Space is a category

- Library contains proof of directed univalence
  $\mathsf{Hom}_{\mathcal{S}}(A, B) \xrightarrow{\sim} (A \to B)$
- There are two holes (Reid's work might help one of these)
- To show $\mathcal{S}$ is Segal, need a version of directed univalence for Δ^2.
  $\mathcal{S}^{\Delta^2} \xrightarrow{\sim} \Sigma_{(A,B,C:\mathcal{S})}(A \to B) \times (B \to C)$
- Have proved a general theorem that covariant families
  $B \times \Delta^1 \to \mathsf{Type}$ correspond to covariant families $P, Q : A \to \mathsf{Type}$
  with a fibrewise map $\Pi_a\,(P(a) \to Q(a))$
- Completeness follows from being Segal and the regular univalence axiom

Conjecture (Gratzer, Weinberger, Schellingerhout, Buchholtz)
Segal types are triangulated.

Corollary
Crisp Segal types are simplicial $\rightsquigarrow$ *simplifies proof for* Space

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More future work

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Cocartesian fibrations

- Basic theory
- Equivalence with (coherent) locally cocartesian fibrations
- Category of categories

Jonathan and Ulrik started some work on this

Thank you